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Queueing with neighbours

Vadim Shcherbakov^a and Stanislav Volkov^b**Abstract**

In this paper we study asymptotic behaviour of a growth process generated by a semi-deterministic variant of the cooperative sequential adsorption model (CSA). This model can also be viewed as a particular example from queueing theory, to which John Kingman has contributed so much. We show that the quite limited randomness of our model still generates a rich collection of possible limiting behaviours.

Keywords cooperative sequential adsorption, interacting particle systems, max-plus algebra, queueing, Tetris

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1 Introduction

Let $\mathcal{M} = \{1, 2, \dots, M\}$ be a lattice segment with periodic boundary conditions (that is, $M + 1$ will be understood as 1 and $1 - 1$ will be understood as M), where $M \geq 1$. The growth process studied in this paper is defined as a discrete-time Markov chain $(\xi_i(t), i \in \mathcal{M}, t \in \mathbb{Z}_+)$,

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with values in \mathbb{Z}_+^M and specified by the following transition probabilities:

$$\begin{aligned} \mathbb{P}(\xi_i(t+1) = \xi_i(t) + 1, \xi_j(t+1) = \xi_j(t) \mid \xi(t)) \\ = \begin{cases} 0, & \text{if } u_i(t) > m(t), \\ 1/N_{\min}(t), & \text{if } u_i(t) = m(t), \end{cases} \end{aligned} \quad (1.1)$$

for $i \in \mathcal{M}$, where

$$u_i(t) = \sum_{j \in U_i} \xi_j(t), \quad i \in \mathcal{M},$$

U_i is a certain neighbourhood of site i ,

$$m(t) = \min_{k \in \mathcal{M}} u_k(t) \quad (1.2)$$

and $N_{\min}(t) \in \{1, 2, \dots, M\}$ is the number of $u_i(t)$ equal to $m(t)$. The quantity $u_i(t)$ is called the *potential* of site i at time t .

The growth process describes the following random sequential allocation procedure. Arriving particles are sequentially allocated at sites of \mathcal{M} such that a particle is allocated uniformly over sites with minimal potential. Then the process component $\xi_k(t)$ is the number of particles at site k at time t . The growth process can be viewed as a certain limit case of a growth process studied in Shcherbakov and Volkov (2009). The growth process in Shcherbakov and Volkov (2009) is defined as a discrete-time Markov chain $(\xi_i(t), i \in \mathcal{M}, t \in \mathbb{Z}_+)$, with values in \mathbb{Z}_+^M and specified by the following transition probabilities

$$\mathbb{P}\{\xi_i(t+1) = \xi_i(t) + 1, \xi_j(t+1) = \xi_j(t) \mid \xi(t)\} = \frac{\beta^{u_i(t)}}{Z(\xi(t))} \quad (1.3)$$

where

$$Z(\xi(t)) = \sum_{j=1}^M \beta^{u_j(t)},$$

β is a positive number and the other notations are the same as before. It is easy to see that the process defined by transition probabilities (1.1) is the corresponding limit process as $\beta \rightarrow 0$. In turn, the growth process specified by the transition probabilities (1.3) is a particular version of the cooperative sequential adsorption model (CSA). CSA is a probabilistic model which is widely used in physics for modelling various adsorption processes (see Evans (1993), Privman (2000) and references therein). Some asymptotic and statistical studies of similar CSA in a continuous setting were undertaken in Shcherbakov (2006), Penrose and Shcherbakov (2009a) and Penrose and Shcherbakov (2009b).

In Shcherbakov and Volkov (2009) we consider the following variants of neighbourhood

- (A1) $U_i = \{i\}$ (empty),
- (A2) $U_i = \{i, i+1\}$ (asymmetric),
- (A3) $U_i = \{i-1, i, i+1\}$ (symmetric),

where, due to the periodic boundary conditions, $U_M = \{M, 1\}$ in case (A2) and $U_1 = \{M, 1, 2\}$, $U_M = \{M-1, M, 1\}$ in case (A3) respectively. It is easy to see that for the growth process studied in this paper the case (A1) is trivial. Therefore, we will consider cases (A2) and (A3) only.

A stochastic process $u(t) = (u_1(t), \dots, u_M(t))$ formed by the sites' potentials plays an important role in our asymptotic study of the growth process. It is easy to see that $u(t)$ is also a Markov chain, with transition probabilities given by

$$\mathbb{P}(u_i(t+1) = u_i(t) + 1_{\{i \in U_k\}}) = \begin{cases} 0, & \text{if } u_k(t) > m(t), \\ 1/N_{\min}(t), & \text{if } u_k(t) = m(t), \end{cases} \quad (1.4)$$

for $k \in \mathcal{M}$. This process has the following rather obvious queueing interpretation (S. Foss, personal communications) explaining the title of the paper (originally titled 'Random sequential adsorption at extremes'). Namely, consider a system with M servers, with the clients arriving in bunches of 2 in case (A2) and of 3 in case (A3). The quantity $u_i(t)$ is interpreted as the number of clients at server i at time t . In case (A2), of the two clients in the arriving pair, one joins the shortest queue, the other its left neighbouring queue, the two choices being equally likely. In case (A3), of the three clients in an arriving triple, one joins the shortest queue, the others its left and right neighbouring queues, the choices being equally likely.

Our goal is to describe the long time behaviour of the growth process, or, equivalently, to describe the limiting queueing profile of the network. It should be noted that the method of proof in this paper is purely combinatorial. This is in contrast with Shcherbakov and Volkov (2009), where the results are proved by combining the martingale techniques from Fayolle et al. (1995) with some probabilistic techniques used in the theory of reinforced processes from Tarrès (2004) and Volkov (2001).

Observe that the model considered here can be viewed as a randomized *Tetris* game, and hence it can possibly be analyzed using the techniques

of max-plus algebra as well; see Bousch and Mairesse (2002) and Section 1.3 of Heidergott et al. (2006) for details.

For the sake of completeness, let us mention another limit case of the growth process specified by transition probabilities (1.3): namely, the limit process arising as $\beta \rightarrow \infty$. It is easy to see that the limit process in this case describes the allocation process in which a particle is allocated with equal probabilities to one of the sites with maximal potential. The asymptotic behaviour (as $t \rightarrow \infty$) of this limit process is briefly discussed in Section 5.

2 Results

Theorem 2.1 *Suppose $U_i = \{i, i+1\}$, $i \in \mathcal{M}$. Then, with probability 1, there is a $t_0 = t_0(\omega)$ (depending also on the initial configuration) such that for all $t \geq t_0$*

$$|\xi_i(t) - \xi_{i+2}(t)| \leq 2 \quad (2.1)$$

for $i \in \mathcal{M}$. Moreover,

$$\xi_i(t) = \frac{t}{M} + \eta_i(t) + \begin{cases} 0, & \text{when } M \text{ is odd,} \\ (-1)^i Z(t), & \text{when } M \text{ is even,} \end{cases} \quad (2.2)$$

where $|\eta_i(t)| \leq 2M$ and for some $\sigma > 0$

$$\lim_{n \rightarrow \infty} \frac{Z(\lfloor sn \rfloor)}{\sigma \sqrt{n}} = B(s),$$

where $\lfloor x \rfloor$ denotes the integer part of x and $B(s)$ is a standard Brownian motion.

Theorem 2.2 *Suppose $U_i = \{i-1, i, i+1\}$, $i \in \mathcal{M}$. Then with probability 1 there exists the limit $\mathbf{x} = \lim_{t \rightarrow \infty} \xi(t)/t$, which takes a finite number of possible values with positive probabilities. The set of limiting configurations consists of those $\mathbf{x} = (x_1, \dots, x_M) \in \mathbb{R}^M$ which simultaneously satisfy the following properties:*

- *there exists an $\alpha > 0$ such that $x_i \in \{0, \alpha/2, \alpha\}$ for all $i \in \mathcal{M}$; also $\sum_{i=1}^M x_i = 1$;*
- *if $x_i = 0$, then $x_{i-1} > 0$ or $x_{i+1} > 0$, or both;*
- *if $x_i = \alpha/2$, then*

$$(x_{j-3}, x_{j-2}, x_{j-1}, x_j, x_{j+1}, x_{j+2}) = (\alpha, 0, \alpha/2, \alpha/2, 0, \alpha),$$

where $j \in \{i, i+1\}$;

- if $x_i = \alpha$, then $x_{i-1} = x_{i+1} = 0$;
- if $M = 3K$ is divisible by 3, then

$$\min\{x_j, x_{j+3}, x_{j+6}, \dots, x_{j+3(K-1)}\} = 0,$$

for $j = 1, 2, 3$.

Moreover, the adsorption eventually stops at all $i \in \mathcal{M}$ where $x_i = 0$, that is

$$\sup_{t \geq 0} \xi_i(t) = \infty \text{ if and only if } x_i > 0.$$

Additionally, if the initial configuration is empty, then for each $x_i = 0$ we must have that **both** $x_{i-1} > 0$ **and** $x_{i+1} > 0$.

Table 2.1 *Limiting configurations for symmetric interaction*

M	Limiting configurations (up to rotation)	No. of limits
4	$(\frac{1}{2}, 0, \frac{1}{2}, 0)$	2
5	$(\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{2}, 0), (\frac{1}{2}, 0, 0, \frac{1}{2}, 0)^*$	5 (10*)
6	$(\frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3}, 0)$	2
7	$(\frac{1}{6}, \frac{1}{6}, 0, \frac{1}{3}, 0, \frac{1}{3}, 0), (\frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3}, 0, 0)^*$	7 (14*)
8	$(\frac{1}{4}, 0, \frac{1}{4}, 0, \frac{1}{4}, 0, \frac{1}{4}, 0),$ $(\frac{1}{6}, \frac{1}{6}, 0, \frac{1}{3}, 0, 0, \frac{1}{3}, 0)^*, (0, 0, \frac{1}{3}, 0, 0, \frac{1}{3}, 0, \frac{1}{3})^*$	2 (18*)
9	$(\frac{1}{8}, \frac{1}{8}, 0, \frac{1}{4}, 0, \frac{1}{4}, 0, \frac{1}{4}, 0), (0, 0, \frac{1}{4}, 0, \frac{1}{4}, 0, \frac{1}{4}, 0, \frac{1}{4})^*$	9 (18*)
10	$(\frac{1}{8}, \frac{1}{8}, 0, \frac{1}{4}, 0, \frac{1}{8}, \frac{1}{8}, 0, \frac{1}{4}, 0), (\frac{1}{5}, 0, \frac{1}{5}, 0, \frac{1}{5}, 0, \frac{1}{5}, 0, \frac{1}{5}, 0),$ $(0, \frac{1}{4}, 0, 0, \frac{1}{4}, 0, \frac{1}{4}, 0, \frac{1}{8}, \frac{1}{8})^*, (0, \frac{1}{4}, 0, \frac{1}{4}, 0, 0, \frac{1}{4}, 0, \frac{1}{8}, \frac{1}{4})^*,$ $(0, 0, \frac{1}{4}, 0, 0, \frac{1}{4}, 0, \frac{1}{4}, 0, \frac{1}{4})^*, (0, 0, \frac{1}{4}, 0, \frac{1}{4}, 0, 0, \frac{1}{4}, 0, \frac{1}{4})^*$	7 (42*)

Table 2.2 *Numbers of limiting configurations for symmetric interaction for larger M*

M	11	12	13	14	15	16
Distinct conf.	1(4*)	2(7*)	1(8*)	3(12*)	2(16*)	3(20*)
All conf.	11(44*)	14(74*)	13(104*)	23(142*)	20(220*)	34(290*)

We will derive the asymptotic behaviour of the process $\xi(t)$ from the asymptotic behaviour of the process of potentials. In turn the study of the process of potentials is greatly facilitated by analysis of the following auxiliary process

$$v_k(t) = u_k(t) - m(t), \quad k = 1, \dots, M. \quad (2.3)$$

Observe that $v(t)$ also forms a Markov chain on $\{0, 1, 2, \dots\}$ and for each t there is a k such that $v_k(t) = 0$. Loosely speaking, the quantities $v_k(t)$, $k = 1, \dots, M$, represent what happens ‘on the top of the growth profile’.

It turns out that in the asymmetric case there is a single class of recurrent states to which the chain eventually falls and then stays in forever. As we show later, this class is a certain subset of the set $\{0, 1, 2\}^M$ containing the origin $(0, \dots, 0)$. Thus a certain ‘stability’ of the process of potentials is observed as time goes to infinity.

In particular, it yields, as we show, that there will not be long queues in the system if M is odd; however, this does not prevent occurrence of relatively long queues if M is even. For instance, if M is even, then one can easily imagine the profile with peaks at even sites, and zeros at odd sites. Besides, observe that the sum of the potentials of the even sites equals the sum of the potentials of the odd sites (see Proposition 3.1); therefore the difference between the total queue to the even sites, and the total queue to the odd ones, behaves similarly to the zero-mean random walk. It means that there are rather long periods of time during which much longer queues are observed at the even sites in comparison with the queues at the odd sites, and vice versa. Thus, in the case of the asymmetric interaction we observe in the limit $t \rightarrow \infty$ a ‘comb pattern’ when M is even, and a ‘flat pattern’ when M is odd.

The picture is completely different in the symmetric case. The Markov chain $v(t)$ is transient for any M ; moreover, there can be only finitely many paths along which the chain escapes to infinity. By this we mean that if the chain reaches a state belonging to a particular *escape path*, then it will never leave it and will go to infinity along this path, and we will show that there can be only a finite number of limit patterns. An escape path can be viewed as *an attractor*, since similar effects are observed in neural network models studied in Karpelevich et al. (1995) and Malyshev and Turova (1997). In fact, the Markov chain $v(t)$ describes the same dynamics as the neural network models in Karpelevich et al. (1995) and Malyshev and Turova (1997) though in a slightly different

set-up. The difference seems to be technical but it results in quite different model behaviour. We do not investigate this issue in depth here.

Table 2.1 contains the list of all possible limiting configurations (for proportions of customers/particles) for small M , while in Table 2 we provide only the numbers of possible limiting configurations for some larger M . Note that in the tables the symbol * stands for the configurations which cannot be achieved from the empty initial configuration. Unfortunately, we cannot compute exact numbers of possible limiting configurations for any M ; nor can we predict which of them will be more likely (though it is obvious that if we start with the empty initial configuration, all possible limits which can be obtained by a rotation of the same \mathbf{x} will have the same probability.)

3 Asymmetric interaction

In the asymmetric case the potential of site k at time t is

$$u_k(t) = \xi_k(t) + \xi_{k+1}(t), \quad k \in \mathcal{M}.$$

The transition probabilities of the Markov chain $u(t) = (u_1(t), \dots, u_M(t))$ are given by

$$\begin{aligned} \mathbb{P}(u_i(t+1) = u_i(t) + 1_{i \in \{k-1, k\}}, i = 1, \dots, M | u(t)) \\ = \begin{cases} 0, & \text{if } u_k(t) > m(t), \\ N_{\min}^{-1}(t), & \text{if } u_k(t) = m(t), \end{cases} \end{aligned}$$

for $k \in \mathcal{M}$, where $N_{\min}(t) \in \{1, 2, \dots, M\}$ is the number of $u_i(t)$ equal to $m(t)$.

Proposition 3.1 *If M is odd, then for any $u = (u_1, u_2, \dots, u_M)$ the system*

$$\begin{aligned} u_1 &= \xi_1 + \xi_2 \\ u_2 &= \xi_2 + \xi_3 \\ &\vdots \\ u_M &= \xi_M + \xi_1 \end{aligned} \tag{3.1}$$

has a unique solution. On the other hand, if M is even, system (3.1) has a solution if and only if

$$u_1 + u_3 + \dots + u_{M-1} = u_2 + u_4 + \dots + u_M. \tag{3.2}$$

Proof For a fixed ξ_1 we can express the remaining ξ_k as

$$\xi_k = u_{k-1} - u_{k-2} + u_{k-3} - \cdots + (-1)^{k-1} \xi_1, \quad (3.3)$$

for any $k = 2, \dots, M$. Now, when M is odd, there is a unique choice of

$$\xi_1 = \frac{1}{2} (u_M - u_{M-1} + u_{M-2} + \cdots - u_2 + u_1).$$

When M is even, by summing separately odd and even lines of (3.1) we obtain condition (3.2). Then it turns out that we can set ξ_1 to be any real number, with ξ_k , $k \geq 2$, given by (3.3). \square

The following statement immediately follows from Proposition 3.1.

Corollary 3.2 *If M is even, then*

$$v_1(t) + v_3(t) + \cdots + v_{M-1}(t) = v_2(t) + v_4(t) + \cdots + v_M(t) \quad \forall t. \quad (3.4)$$

In the following two Propositions we will show that when either M is odd or M is even and condition (3.4) holds, the state $\mathbf{0} = (0, \dots, 0)$ is recurrent for the Markov chain $v(t)$. First, define the following stopping times

$$\begin{aligned} t_0 &= 0, \\ t_j &= \min\{t > t_{j-1} : m(t_j) > m(t_{j-1})\}, \quad j = 1, 2, \dots \end{aligned} \quad (3.5)$$

Let

$$S(j) = \sum_{k=1}^M v_k^*(t_j)$$

where

$$v_k^*(t_j) = \begin{cases} v_k(t_j), & \text{if } v_k(t_j) \geq 2, \\ 0, & \text{otherwise,} \end{cases}$$

where the stopping times are defined by (3.5).

Proposition 3.3 *$S(j+1) \leq S(j)$. Moreover, there is an integer $K = K(M)$ and an $\varepsilon > 0$ such that*

$$\mathbb{P}(S(j+K) - S(j) \leq -1 \mid v(t_j)) \geq \varepsilon$$

on the event $S(j) > 0$.

Proof For simplicity, let us write v_k for $v_k(t_j)$. Take some non-zero

element $a \geq 1$ in the sequence of v_k at time t_j . Whenever it is followed by a consecutive chunk of 0s, namely

$$\dots a \underbrace{0 \ 0 \ \dots \ 0}_{m} \dots$$

at time t_{j+1} this becomes either

$$\dots a \underbrace{z_1 \ z_2 \ \dots \ z_m}_{m} \dots$$

or

$$\dots a - 1 \underbrace{z_1 \ z_2 \ \dots \ z_m}_{m} \dots,$$

where $z_i \in \{0, 1\}$, and the latter occurs if the second 0 from the left is chosen before the first one. On the other hand, if a is succeeded by a non-zero element, say ' $\dots a \ b \ \dots$ ' at time t_{j+1} this becomes either ' $\dots a - 1 \ b \ \dots$ ' or ' $\dots a - 1 \ b - 1 \ \dots$ '. In all cases, this leads to $S(j+1) \leq S(j)$.

Secondly, from the previous arguments we see that if there is at least one $a \geq 2$ in the sequence of $(v_1 \dots v_M)$ followed by a non-zero element, then this element becomes $a - 1$ at t_{j+1} and hence $S(j+1) \leq S(j) - 1$.

Now let us investigate what happens if the opposite occurs. Then each element $a \geq 2$ in $(v_1 \dots v_M)$ is followed by a sequence of 0s and 1s such that we observe either

$$\dots a \ 0 \ b \dots$$

or

$$\dots a \ 0 \ 1 \underbrace{z_3 \ z_4 \ \dots \ z_m}_{m-2} \ b \dots,$$

where $m \geq 2$, $b \geq 2$ and $z_i \in \{0, 1\}$. Because of Corollary 3.2, we cannot have an alternating sequence of 0s and non-zero elements; therefore, we must be able to find somewhere in the sequence of vs a chunk which looks either like

$$\dots a \underbrace{0 \ 1 \ 0 \ 1 \ \dots \ 0 \ 1}_{2l \text{ elements}} \ 0 \ 1 \ c \dots \text{ where } c \geq 1 \quad (\text{A})$$

or

$$\dots a \underbrace{0 \ 1 \ 0 \ 1 \ \dots \ 0 \ 1}_{2l \text{ elements}} \underbrace{0 \ 0 \ ?}_{i \ i+1 \ i+2} \dots \quad (\text{B})$$

where $l \geq 0$. Note that a configuration of type (A) at time t_{j+1} with probability 1 becomes a configuration of type (B). At the same time, in configuration (B), with probability of at least $\frac{1}{3}$ the 0 located at position $i+1$ is chosen before either the 0 at position i or (possibly) the 0 at position $i+2$. On this event, the configuration in (B) at time t_{j+1} becomes

$$\dots a \underbrace{0\ 1\ 0\ 1\ \dots\ 0\ 1}_{2l-2 \text{ elements}} 0\ 0\ 0\ ?\dots \quad (\text{B}')$$

By iterating this argument until $l = 0$, we conclude that eventually there will be a chunk ' $\dots a\ 0\ 0\dots$ ' on some step $t_{j'}$ which in turn at time $t_{j'+1}$ will become ' $\dots a - 1\ 0\ ?\dots$ ' with probability at least $\frac{1}{3}$, resulting in $S_{j'+1} \leq S_{j'} - 1$. This yields the statement of Proposition 3.3 with $K = M$ and $\varepsilon = 3^{-M}$. \square

Proposition 3.4 *With probability 1, there is a $j_0 = j_0(\omega)$ such that*

$$S(j) = 0 \text{ for all } j \geq j_0.$$

Additionally, the state $\mathbf{0} = (0, 0, \dots, 0)$ is recurrent for the Markov chain $v(t)$.

Proof The first statement trivially follows from Proposition 3.3. Next observe that at times $t_j \geq t_{j_0}$ the sequence $(v_1(t_j), \dots, v_M(t_j))$ consists only of 0s and 1s locally looking either like

$$\dots 1 \underbrace{0\ 0\ 0\ 0\ \dots\ 0\ 0}_{2l \text{ elements}} 1\dots \quad (\text{C})$$

or

$$\dots 1 \underbrace{0\ 0\ 0\ 0\ \dots\ 0\ 0}_{2l \text{ elements}} 0\ 1\dots \quad (\text{D})$$

With positive probability even-located 0s are picked before odd-located 0s, hence at time t_{j+1} configuration (C) becomes

$$\dots 0 \underbrace{0\ 0\ 0\ 0\ \dots\ 0\ 0}_{2l \text{ elements}} ?\dots \quad (\text{C}')$$

while configuration (D) becomes

$$\dots 0 \underbrace{0\ 0\ 0\ 0\ \dots\ 0\ 0}_{2l-2 \text{ elements}} 0\ 1\ 0\ ?\dots \quad (\text{D}')$$

In both cases (C) \rightarrow (C') and (D) \rightarrow (D') the number of 1s among the v_i does not increase, and in the first case it goes down by 1. However, it is easy to see that whether M is odd or even (in the latter case due to Corollary 3.2) there will be at least one chunk of type (C), and hence with positive probability $v(t)$ reaches state $\mathbf{0}$ in at most M^2 steps (since $t_{j+1} - t_j \leq M$). The observation that after t_{j_0} the Markov chain $v(t)$ lives on a finite state space $\{0, 1, 2\}^M$ finishes the proof. \square

Proof of Theorem 2.1 The first part easily follows from Proposition 3.4 and the definition of potentials v . Indeed, for $j \geq j_0$ and all i we have $v_i(t_j) \in \{0, 1\}$, while for $t \in (t_j, t_{j+1})$ we have $v_i(t) \in \{0, 1, 2\}$. On the other hand, omitting (t) , we can write $v_{i+1} - v_i = u_{i+1} - u_i = \xi_{i+2} - \xi_i$, $i \in \mathcal{M}$, yielding (2.1).

Next, iterating this argument, we obtain $|\xi_{i+2l} - \xi_i| \leq 2l$. Because of the periodic boundary condition, in the case when M is odd, this results in $|\xi_i - \xi_j| \leq 2M$ for all i and j , while in the case when M is even this is true only whenever $i - j$ is even. The observation that $\sum_j \xi_j(t) = t$ thus proves (2.2) for odd M , since

$$|t - M\xi_i(t)| = \left| \sum_{j=1}^M [\xi_j(t) - \xi_i(t)] \right| \leq (M-1) \times 2M.$$

Now, when $M = 2L$ is even, denote

$$H(t) = \frac{\sum_{j=1}^L \xi_{2j}(t) - \sum_{j=1}^L \xi_{2j-1}(t)}{M}.$$

Suppose $i \in \mathcal{M}$ is even. Then

$$\begin{aligned} |t - M\xi_i(t) + MH(t)| &= \left| 2 \sum_{j=1}^L \xi_{2j}(t) - 2L\xi_i(t) \right| \\ &= \left| 2 \sum_{j=1}^L [\xi_{2j}(t) - \xi_i(t)] \right| \leq 4M(L-1) < 2M^2. \end{aligned}$$

A similar argument holds for odd i . Hence we have established (2.2) for even M as well as for odd M .

To finish the proof, denote by τ_m , $m \geq 0$, the consecutive renewal times of the Markov chain $v(t)$ after t_{j_0} , that is

$$\begin{aligned} \tau_0 &= \inf\{t \geq t_{j_0} : v_1(t) = v_2(t) = \dots = v_M(t) = 0\}, \\ \tau_m &= \inf\{t \geq \tau_{m-1} : v_1(t) = v_2(t) = \dots = v_M(t) = 0\}, \quad m \geq 1. \end{aligned}$$

By Proposition 3.4, these stopping times are well-defined; moreover, $\tau_{m+1} - \tau_m$ are i.i.d. and have exponential tails. Let $\zeta_{m+1} = H(\tau_{m+1}) - H(\tau_m)$. Then the ζ_m are also i.i.d.; moreover, their distribution is symmetric around 0, and $|\zeta_{m+1}| \leq \tau_{m+1} - \tau_m$ hence the ζ_m also have exponential tails. The rest follows from the standard Donsker–Varadhan invariance principle; see e.g. Durrett et al. (2002), pp. 590–592, for a proof in a very similar set-up. \square

4 Symmetric interaction

In the symmetric case, the potential of site k at time t is

$$u_k(t) = \xi_{k-1}(t) + \xi_k(t) + \xi_{k+1}(t), \quad k \in \mathcal{M}, \quad (4.1)$$

and the transition probabilities of the Markov chain $u(t)$ are now given by

$$\begin{aligned} \mathbb{P}(u_i(t+1) = u_i(t) + 1_{i \in \{k-1, k, k+1\}}, i = 1, \dots, M | u(t)) \\ = \begin{cases} 0, & \text{if } u_k(t) > m(t), \\ N_{\min}^{-1}(t), & \text{if } u_k(t) = m(t), \end{cases} \end{aligned}$$

for $k \in \mathcal{M}$, where, as before, $N_{\min}(t) \in \{1, 2, \dots, M\}$ is the number of $u_i(t)$ equal to $m(t)$.

Proposition 4.1 *If $(M \bmod 3) \neq 0$, then for any $u = (u_1, u_2, \dots, u_M)$ the system*

$$\begin{aligned} u_1 &= \xi_M + \xi_1 + \xi_2 \\ u_2 &= \xi_1 + \xi_2 + \xi_3 \\ &\vdots \\ u_M &= \xi_{M-1} + \xi_M + \xi_1 \end{aligned} \quad (4.2)$$

has a unique solution. On the other hand, if M is divisible by 3, system (4.2) has a solution if and only if

$$\begin{aligned} u_1 + u_4 + \dots + u_{M-2} &= u_2 + u_5 + \dots + u_{M-1} \\ &= u_3 + u_6 + \dots + u_M. \end{aligned} \quad (4.3)$$

Proof If M is not divisible by 3, then the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix}$$

corresponding to the equation (4.2) is ± 3 (which can be easily proved by induction). Hence the system must have a unique solution.

When M is divisible by 3, by summing separately the 1st, 4th, 5th, ... lines of (4.2), and then repeating this for the 2nd, 5th, ... or 3th, 6th, ... lines, we obtain condition (4.3).

Then it turns out that we can set both ξ_1 and ξ_2 to be any real numbers, so $\xi_3 = u_2 - \xi_1 - \xi_2$, and ξ_k , $k \geq 4$, are given:

$$\xi_{k+1} = [u_k - u_{k-1}] + [u_{k-3} - u_{k-4}] + \dots + \xi_{(k \bmod 3)+1}.$$

□

Similarly to the asymmetric case, consider the Markov chain $v(t)$ on $\{0, 1, 2, \dots\}$ and recall the definition of t_j from (2.3). The following statement is straightforward.

Proposition 4.2 *For any $k \in \mathcal{M}$*

$$v_k(t_{j+1}) \leq v_k(t_j)$$

unless both $v_{k-1}(t_j) = 0$ and $v_{k+1}(t_j) = 0$.

Proposition 4.3 *For j large enough, in the sequence of $v_k(t_j)$, $k \in \mathcal{M}$, there are no more than two non-zero elements in a row, that is*

if $v_k(t_j) > 0$ then either $v_{k-1}(t_j) = 0$ or $v_{k+1}(t_j) = 0$, or both.

Proof Fix some $k \in \mathcal{M}$. Then $v_k(t_j)$ is either 0 or positive. In the first case, unless both of the neighbours of point k are zeros at time t_j , by Proposition 4.2 we have $v_k(t_{j+1}) = 0$. On the other hand, if $(v_{k-1}(t_j), v_k(t_j), v_{k+1}(t_j)) = (0, 0, 0)$, then at time t_{j+1} either this triple becomes $(0, 1, 0)$ if both $k-1$ and $k+1$ are chosen, or $v_k(t_{j+1}) = 0$.

Now suppose that $v_k(t_j) > 0$. If both $v_{k-1}(t_j) = 0$ and $v_{k+1}(t_j) = 0$, then from Proposition 4.2 applied to $k-1$ and $k+1$, we conclude $v_{k-1}(t_{j+1}) = v_{k+1}(t_{j+1}) = 0$, hence point k remains surrounded by 0s.

Similarly, if $v_k(t_j) > 0$ and $v_{k+1}(t_j) > 0$ but $v_{k-1}(t_j) = v_{k+2}(t_j) = 0$, then points $\{k, k+1\}$ remain surrounded by 0s at time t_{j+1} .

Finally, if point k is surrounded by non-zeros on both sides, that is $v_{k-1}(t_j)$, $v_k(t_j)$ and $v_{k+1}(t_j)$ are all positive, we have $v_k(t_{j+1}) = v_k(t_j) - 1$.

Consequently, all sequences of non-zero elements of length ≥ 3 are bound to disappear, and no such new sequence can arise as j increases. \square

Proposition 4.4 *For any $k \in \mathcal{M}$, if for some s*

$$v_{k-1}(s) > 0, \quad v_k(s) = 0, \quad v_{k+1}(s) > 0$$

then for all j such that $t_j \geq s$

$$v_{k-1}(t_j) > 0, \quad v_k(t_j) = 0, \quad v_{k+1}(t_j) > 0.$$

Proof This immediately follows from the fact that there must be a particle adsorbed at point k during the time interval $(t_{j_0}, t_{j_0+1}]$ where $j_0 = \max\{j : t_j \leq s\}$, and that would imply that $v_{k\pm 1}(t_{j_0+1}) \geq v_{k\pm 1}(t_{j_0})$ while $v_k(t_{j_0+1}) = 0$. Now an induction on j finishes the proof. \square

Proposition 4.5 *For j large enough, in the sequence of $v_k(t_j)$, $k \in \mathcal{M}$, there are no more than two 0s in a row, that is*

$$\text{if } v_k(t_j) = 0 \text{ then either } v_{k-1}(t_j) > 0 \text{ or } v_{k+1}(t_j) > 0, \text{ or both.}$$

Proof Suppose j is so large that already there are no consecutive subsequences of positive elements of length ≥ 2 in (v_1, \dots, v_M) (see Proposition 4.3). Let

$$Q(j) = |\{k : v_{k-1}(t_j) > 0, v_k(t_j) = 0, v_{k+1}(t_j) > 0\}|.$$

Proposition 4.4 implies that $Q(j)$ is non-decreasing; since $Q(j) < M$ it means that $Q(j)$ must converge to a finite limit.

Let A_j be the event that at time t_j there are 3 or more zeroes in a row in $v(t_j)$. On A_j there is a $k \in \mathcal{M}$ such that $v_k(t_j) = v_{k+1}(t_j) = v_{k+2}(t_j) = 0$ but $v_{k-1}(t_j) > 0$, (unless all $v_k = 0$ but then the argument is similar). Then, with a probability exceeding $1/M$, at time $t_j + 1$ new particle gets adsorbed at $k + 2$, yielding by Proposition 4.4 that for all $j' > j$ we have $v_{k-1}(t_{j'}) > 0$, $v_k(t_{j'}) = 0$, $v_{k+1}(t_{j'}) > 0$, hence the event $B_j := \{Q(j+1) \geq Q(j) + 1\}$ occurs as well. Therefore,

$$\mathbb{P}(B_j | \mathcal{F}_{t_j}) \geq \frac{1}{M} \mathbb{P}(A_j | \mathcal{F}_{t_j}),$$

where \mathcal{F}_{t_j} denotes the sigma-algebra generated by $v(t)$ by time t_j . Combining this with the second Borel–Cantelli lemma, we obtain

$$\begin{aligned} \{A_j \text{ i.o.}\} &= \left\{ \sum_j \mathbb{P}(A_j | \mathcal{F}_{t_j}) = \infty \right\} \subseteq \left\{ \sum_j \mathbb{P}(B_j | \mathcal{F}_{t_j}) = \infty \right\} \\ &= \{B_j \text{ i.o.}\} = \{Q(j) \rightarrow \infty\} \end{aligned}$$

leading to a contradiction. \square

Proposition 4.6 *Let*

$$W(j) = |\{k : v_{k-1}(t_j) = 0, v_k(t_j) > 0, v_{k+1}(t_j) > 0, v_{k-1}(t_j) = 0, \}|$$

be the number of ‘doubles’. Then $W(j)$ is non-increasing.

Proof Let us investigate how we can obtain a subsequence $(0, *, *, 0)$ starting at position $k - 1$ at time t_{j+1} , where $*$ stands for a positive element. One possibility is that at time t_j we already have such a subsequence there; this does not increase $W(j)$. The other possibilities at time t_j are

$$\begin{aligned} &(0, 0, 0, 0), (0, 0, 0, *), (0, 0, *, 0), (0, 0, *, *), \\ &(0, *, 0, 0), (0, *, 0, *), (*, *, 0, 0), (*, *, 0, *). \end{aligned}$$

By careful examination of all of the configurations above, we conclude that the subsequence $(0, *, *, 0)$ cannot arise at time t_{j+1} . Consequently, $W(j)$ cannot increase. \square

Proposition 4.7 *For j large enough, in the sequence of $v_k(t_j)$, $k \in \mathcal{M}$, there are no consecutive subsequences of the form $(*, *, 0, 0)$ or $(0, 0, *, *)$ where each $*$ stands for any positive number; that is there is no k such that*

$$v_k(t_j) = v_{k+1}(t_j) = 0 \text{ and either}$$

$$v_{k+2}(t_j) > 0 \text{ and } v_{k+3}(t_j) > 0$$

$$\text{or } v_{k-1}(t_j) > 0 \text{ and } v_{k-2}(t_j) > 0.$$

Proof Omitting (t_j) , without loss of generality suppose $v_k > 0$, $v_{k+1} > 0$, $v_{k+2} = v_{k+3} = 0$. Then either at some time $j_1 > j$ we will have $v_{k+3}(t_{j_1}) > 0$ (hence the configuration $(*, *, 0, 0)$ gets destroyed), or with probability at least $\frac{1}{3}$ for each $j' \geq j$ we have adsorption at position $k+3$ at some time during the time interval $(t_{j'}, t_{j'+1}]$. This would imply that $v_{k+1}(t_{j'+1}) = v_{k+1}(t_{j'}) - 1$. Hence, in a geometrically distributed number of times, we obtain 0 at position $k+1$, and thus the configuration

$(*, *, 0, 0)$ gets destroyed. On the other hand, by Proposition 4.6, the number of doubles is non-increasing, so no new configurations of this type can arise. Consequently, eventually all configurations $(*, *, 0, 0)$ and $(0, 0, *, *)$ will disappear. \square

Proposition 4.8 *For j large enough, in the sequence of $v_k(t_j)$, $k \in \mathcal{M}$, there are no consecutive subsequences of the form $(0, 0, *, 0, 0)$ where $*$ stands for any positive number; that is there is no k such that*

$$v_{k-2}(t_j) = v_{k-1}(t_j) = 0 = v_{k+1}(t_j) = v_{k+2}(t_j) \text{ and } v_k(t_j) > 0.$$

Proof Propositions 4.3 and 4.5 imply that for some (random) J large enough for all $j \geq J$ consecutive subsequences of zero (non-zero resp.) elements have length ≤ 2 , and Proposition 4.7 says that two consecutive 0s must be followed (preceded resp.) by a single non-zero element. Therefore, $(0, 0, *, 0, 0)$ must be a part of a longer subsequence of form $(0, *, 0, 0, *, 0, 0, *, 0)$. This, in turn, implies for the middle non-zero element located at k that

$$v_k(t_{j+1}) = \begin{cases} v_k(t_j) + 1, & \text{with probability } 1/4, \\ v_k(t_j), & \text{with probability } 1/2, \\ v_k(t_j) - 1, & \text{with probability } 1/4. \end{cases}$$

Hence, by the properties of simple random walk, for some $j' > J$ we will have $v_k(t_{j'}) = 0$ (suppose that j' is the first such time). On the other hand, by Proposition 4.2,

$$v_{k-2}(t_{j'}) = v_{k-1}(t_{j'}) = v_{k+1}(t_{j'}) = v_{k+2}(t_{j'}) = 0$$

as well. This yields a contradiction with the choice of J (see Proposition 4.5). \square

Proof of Theorem 2.2 Let a *configuration of the potential* be a sequence $\bar{v} = (\bar{v}_1, \dots, \bar{v}_M)$ where each $\bar{v}_i \in \{0, *\}$. Then we say that $v = (v_1, v_2, \dots, v_M)$ with the following property has type \bar{v} :

$$\begin{aligned} v_i &= 0 \text{ if } \bar{v}_i = 0, \\ v_i &> 0 \text{ if } \bar{v}_i = *. \end{aligned}$$

Propositions 4.3, 4.5, 4.7, and 4.8 rule out various types of configurations for all j large enough. On the other hand, it is easy to check that all remaining configurations for $v(t_j)$ are possible and stable, that is, once you reach them, you stay in them forever.

Call a configuration \bar{v} *admissible*, if there is a collection $\xi_1, \xi_2, \dots, \xi_M$ such that the system (4.2) has a solution for some $u = (u_1, \dots, u_M)$

having type \bar{v} . If M is not divisible by 3, according to Proposition 4.2 all configurations \bar{v} are admissible. On the other hand, it is easy to see that if $M = 3K$ then a necessary and sufficient condition for a non-zero configuration \bar{v} to be admissible is

$$\begin{aligned}\bar{v}_i &= * \quad \text{for some } i \text{ such that } i \bmod 3 = 0, \text{ and} \\ \bar{v}_j &= * \quad \text{for some } j \text{ such that } j \bmod 3 = 1, \text{ and} \\ \bar{v}_k &= * \quad \text{for some } k \text{ such that } k \bmod 3 = 2.\end{aligned}$$

This establishes all possible stable configurations for v and hence the potential u , thus determining the subset of \mathcal{M} where points are adsorbed for sufficiently large times, namely, $\xi_i(t) \rightarrow \infty$ if and only if $v_i(t_j) = 0$ for all large j .

Moreover, whenever we see a subsequence of type $(v_{k-1}, v_k, v_{k+1}) = (*, 0, *)$, we have

$$0 \leq \lim_{j \rightarrow \infty} [t_j - u_k(t_j)] < \infty,$$

and for a subsequence of type $(v_{k-1}, v_k, v_{k+1}, v_{k+2}) = (*, 0, 0, *)$ we have

$$\lim_{j \rightarrow \infty} \frac{u_k(t_j)}{t_j} = \lim_{j \rightarrow \infty} \frac{u_{k+1}(t_j)}{t_j} = \frac{1}{2}$$

by the strong law. Setting

$$\alpha = \frac{1}{\lim_{j \rightarrow \infty} |\{i \in \mathcal{M} : v_i(t_j) > 0, v_{i+1}(t_j) = 0\}|}$$

finishes the proof of the first part of the Theorem.

Finally, note that if the initial configuration is empty, the conditions of Proposition 4.6 are fulfilled with no ‘doubles’ at all, i.e. $W(0) = 0$. Consequently, for all $j \geq 0$ we have that there are no consecutive non-zero elements in $v_k(t_j)$, yielding the final statement of the Theorem. \square

5 Appendix

In this section we briefly describe the long-time behaviour of the growth process generated by the dynamics, where a particle is allocated at random to a site with maximum potential. The process is trivial in both the symmetric and asymmetric cases. Consider the symmetric case, i.e. $U_i = \{i-1, i, i+1\}$, $i \in \mathcal{M}$. It is easy to see that with probability 1,

there exists k such that either

$$\lim_{t \rightarrow \infty} \frac{\xi_k(t)}{t} = 1 \quad \text{and} \quad \sup_{i \neq k} \xi_i(t) < \infty \quad (5.1)$$

or

$$\lim_{t \rightarrow \infty} \frac{\xi_k(t)}{t} = \lim_{t \rightarrow \infty} \frac{\xi_{k+1}(t)}{t} = \frac{1}{2} \quad \text{and} \quad \sup_{i \notin \{k, k+1\}} \xi_i(t) < \infty. \quad (5.2)$$

Indeed, recall the formula for the potential given by (4.1). Then $u(t)$ is a Markov chain with transition probabilities given by

$$\mathbb{P}(u_i(t+1) = u_i(t) + 1_{i \in \{k-1, k, k+1\}}) = \frac{1_{\{k \in S_{\max}(t)\}}}{|S_{\max}(t)|}$$

for $k \in \mathcal{M}$, where

$$S_{\max} = \left\{ i : u_i(t) = \max_{i \in \mathcal{M}} u_i(t) \right\} \subseteq \mathcal{M}$$

is the set of those i for which $u_i(t)$ equals the maximum value.

Observe that if at time s the adsorption/allocation occurs at point i , then $S_{\max}(s+1) \subseteq \{i-1, i, i+1\}$. In particular, if the maximum is unique, that is, $S_{\max}(s+1) = \{i\}$, then for all times $t \geq s$ this property will hold, and hence all the particles from now on will be adsorbed at i only.

If, on the other hand, $|S_{\max}(s+1)| = 2$, without loss of generality say $S_{\max}(s+1) = \{i, i+1\}$, then this property will be also preserved for all $t > s$ and each new particle will be adsorbed with probability $\frac{1}{2}$ at either i or $i+1$.

Finally, if $|S_{\max}(s+1)| = 3$, say $S_{\max}(s+1) = \{i, i+1, i+2\}$, then at time $s+2$ either $S_{\max}(s+2) = \{i, i+1, i+2\}$ if the adsorption occurred at $i+1$, or $S_{\max}(s+2) = \{i+1, i+2\}$ or $\{i, i+1\}$ otherwise. By iterating this argument we obtain that after a geometric number of times we will arrive at the situation where $|S_{\max}(t)| = 2$, and then the process will follow the pattern described in the previous paragraph.

A similar simple argument shows that in the case of the asymmetric interaction only the outcome (5.1) is possible.

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